

Exact solution for the critical state in thin superconductor strips with field dependent or anisotropic pinning

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(February 1, 2008)

An exact analytical solution is given for the critical state problem in long thin superconductor strips in a perpendicular magnetic field, when the critical current density $j_c(B)$ depends on the local induction B according to a simple three-parameter model. This model describes both isotropic superconductors with this $j_c(B)$ dependence, but also superconductors with anisotropic pinning described by a dependence $j_c(\theta)$ where θ is the tilt angle of the flux lines away from the normal to the specimen plane.

PACS numbers: **74.60.-w**, **74.60.Ge**, **74.60.Jg**

I. INTRODUCTION

The critical state model¹ for the magnetic behavior of superconductors with flux-line pinning has proven very useful² though it originally was applied to the simple (demagnetization-free) *longitudinal* geometry of long superconductors in parallel magnetic field. It took over 30 years until an analytical solution of the critical state model was obtained for the more realistic *transverse* geometry of thin superconductors. The solutions were derived for thin disks³ and strips⁴ in a perpendicular magnetic field, extending an earlier work on superconductor strips with transport current,⁵ and finally for elliptic-shaped platelets.⁶ Recent detailed numerical work for strips⁷ and disks⁸ of finite thickness shows how the transition from longitudinal to transverse geometry occurs with changing aspect ratio of the specimen.

So far, in the transverse geometry all analytical solutions of the critical state model were restricted to the Bean model of constant critical current density $j_c = \text{const}$, but in many experiments $j_c = j_c(B)$ depends on the local magnetic induction B . For example, the simple Kim model⁹ $j_c(B) = j_c(0)/(1 + |B|/B_0)$ was considered in many experimental and theoretical papers, see e.g. the reviews 2,10,11 and the partly analytical calculations for thin strips¹² and disks¹³. While numerical computations easily allow us to consider any $j_c(B)$ dependence,^{7,8,11,14} an exact analytical solution of some model may give deeper insight since it yields explicit dependences of the resulting quantities on the input parameters.

In the highly anisotropic high- T_c superconductors the flux-line pinning in general depends on the angle θ between the local direction of the magnetic induction \mathbf{B} and the c axis, which in typical experiments is normal to the plane of the sample. For example, this type of anisotropy occurs when one takes into account the intrinsic pinning exerted by the CuO planes or the pinning

by extended defects.¹¹ It has been shown recently^{15–18} that for thin superconductors of any shape (with thickness d much smaller than the lateral extension L but larger than the magnetic penetration depth λ) any such out-of-plane-anisotropy of pinning is equivalent to an induction dependence of the critical sheet current $J_c(B)$ (the sheet current is defined as the current density integrated over the film thickness). Thus, the description of the two-dimensional critical state, e.g., in an anisotropic strip can be reduced to the analysis of a one-dimensional problem with some $J_c(B)$. In this case the characteristic scale B_0 over which $J_c(B)$ changes is of the order of $\mu_0 j_c d$.

In this paper we present an analytical solution for the critical state in thin superconductor strips in perpendicular field with field dependent critical current density $j_c(B)$ or, equivalently, with anisotropic pinning described by a $j_c(\theta)$. A three-parameter model $j_c(B)$ consisting of two straight lines, an inclined line at small B and a horizontal line at larger B , is considered. This rather general model is equivalent to a piecewise constant angular dependence $j_c(\theta) = j_{c1}$ for $0 \leq \theta < \theta_0$ and $j_c(\theta) = j_{c2}$ for $\theta_0 \leq \theta < \pi/2$ where j_{c1} , j_{c2} , and θ_0 are the parameters of the model. We shall show below that the steepness of the flux front in the superconductor essentially depends on the anisotropy of pinning. In particular, in the case corresponding to the intrinsic pinning in high- T_c superconductors, the front is a very sharp step, which should be taken into account in analyzing data of local magnetic measurements. We shall also show that under certain conditions two penetrating flux fronts can occur in an anisotropic superconductor.

As usual, we consider here the cases when the characteristic magnetic field in the sample is sufficiently large such that the difference between the magnetic induction B and the field H may be disregarded. This condition is satisfied when $j_c d$ is much larger than the lower criti-

cal field H_{c1} (otherwise, the so-called geometric barrier¹⁹ must be taken into account). We shall thus express all the following equations in terms of the magnetic field H , related to the current density by the Maxwell equation $\mathbf{j} = \nabla \times \mathbf{H}$.

II. MODEL AND ITS SOLUTION

We consider an infinitely long strip of width $2w$ and thickness d , filling the space $-w \leq x \leq w$, $-d/2 \leq z \leq d/2$, i.e. we place the y axis of the coordinate system along the central line of the strip and the z axis along the external magnetic field H_a which is applied normal to the plane of the strip. The increasing applied field induces a sheet current J along y , which is related to the z component of the magnetic field in the plane $z = 0$ by the Biot-Savart law,

$$H_z(x) = H_a + \frac{1}{2\pi} \int_{-w}^w \frac{J(t) dt}{t-x}. \quad (1)$$

Here and below all singular integrals are taken in the sense of the Cauchy principal value. The penetration of the magnetic flux into the superconducting strip is described by the following critical state equations: In the flux-free central region $|x| \leq b(H_a)$ one has

$$H_z = 0, \quad (2)$$

while in the region $b(H_a) \leq |x| \leq w$, where the flux already exists, one has

$$|J(x)| = J_c[H_z(x)]. \quad (3)$$

The position $x = b(H_a)$ of the boundary separating the regions, is found by solving these equations. In Eq. (3) $J_c(H_z)$ is the critical value of the sheet current. At present an exact solution of Eqs. (1)–(3) is known^{4,20,21} only for the Bean critical state model where $J_c = \text{const}$. Below we shall obtain the exact solution for the more general case when $J_c(|H_z|)$ has the model form (see Fig. 1):

$$\begin{aligned} J_c(H_z) &= J_{c1} - \gamma H_z & \text{for } 0 \leq H_z \leq H_z^0, \\ J_c(H_z) &= J_{c0} & \text{for } H_z \geq H_z^0. \end{aligned} \quad (4)$$

Here $\gamma = (J_{c1} - J_{c0})/H_z^0$; the three parameters J_{c1} , J_{c0} , and H_z^0 may have any positive value.

As was mentioned above, in the case of thin superconductors the dependence of the critical current density j_c on the angle θ between the local direction of the magnetic induction and the normal to the strip plane can be taken into account if one considers this superconductor as infinitely thin but with an H_z dependent sheet current. The model dependence described by Eqs. (4) corresponds to the following θ -dependence of the critical current density¹⁸ shown in Fig. 1:

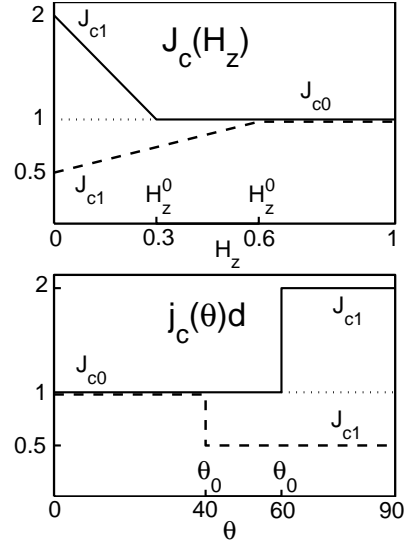


FIG. 1. Visualization of the dependence of the critical sheet current on the perpendicular magnetic field [$J_c(H_z)$, Eq. (4), upper plot], equivalent to an out-of-plane anisotropy [$j_c(\theta)$, Eq. (5), lower plot]. The model has three independent positive parameters, J_{c0} , J_{c1} , and H_z^0 , all of same dimension. In this plot we put $J_{c0} = 1$ and show two examples: $J_{c1} = 2$ (intrinsic pinning, solid lines) and $J_{c1} = 0.5$ (dashed lines), with $H_z^0 = 0.3$ (0.6) equivalent to $\theta_0 = \arctan(J_{c0}/2H_z^0) \approx 60$ (40) degrees.

$$\begin{aligned} j_c(\theta) &= J_{c0}/d & \text{for } 0 \leq \theta \leq \theta_0, \\ j_c(\theta) &= J_{c1}/d & \text{for } \theta_0 \leq \theta \leq \pi/2, \end{aligned} \quad (5)$$

where $\tan \theta_0 = J_{c0}/2H_z^0$. Thus, the case $\gamma > 0$ describes intrinsic pinning by the CuO planes in high- T_c superconductors²² (j_c peaks at $\theta = \pi/2$), whereas the case $\gamma < 0$ can be used to analyze pinning by columnar defects normal to the film (j_c peaks at $\theta = 0$). In both these cases one can find *two-dimensional* solutions of the critical state equations for strips of small but finite thickness using the results obtained below and Eqs. (5,6,9–11) of Ref. 18.

Accounting for the symmetry of the sheet current, $J(-x) = -J(x)$, we seek the solution of Eqs. (1)–(4) in the form

$$J(x) = -\frac{x}{|x|} [J_0(x) + J_1(x)] \quad (6)$$

where

$$J_0(x) = J_{c0}, \quad b^2 \leq x^2 \leq w^2, \quad (7)$$

$$J_0(x) = \frac{2J_{c0}}{\pi} \arctan \left[\frac{(w^2 - b^2)x^2}{w^2(b^2 - x^2)} \right]^{1/2}, \quad x^2 \leq b^2, \quad (8)$$

while $J_1(x)$ is a new unknown function. The parameter b defines the position of the flux front, i.e., $x = b$ is the point where H_z goes to zero. This parameter depends on H_a and must be determined together with $J_1(x)$. Both $J_0(x)$ and $J_1(x)$ (and the magnetic field below) are even

functions, which depend only on x^2 . The function $J_0(x)$ has the form of the exact solution⁴ to Eqs. (1)–(3) in the case when $J_c = J_{c0}$ and the external magnetic field is equal to

$$H_b = H_{cs} \operatorname{arccosh}(w/b)$$

where $H_{cs} = J_{c0}/\pi$. Using Eqs. (1), (6)–(8), the expression for the magnetic field can be rewritten as

$$H_z(x) = H_0(x) - \frac{1}{2\pi} \int_0^{a^2} \frac{J_1(\sqrt{s}) ds}{s - x^2}, \quad (9)$$

where a is defined by the equality $H_z(a) = H_z^0$, and $H_0(x)$ is the sum of H_a and the field generated by the current $J_0(x)$,⁴

$$H_0(x) = H_a - H_b, \quad 0 \leq x^2 \leq b^2, \quad (10)$$

$$H_0(x) = H_a - H_b +$$

$$H_{cs} \operatorname{arctanh} \left[\frac{(x^2 - b^2) w^2}{x^2 (w^2 - b^2)} \right]^{1/2}, \quad b^2 \leq x^2 \leq w^2. \quad (11)$$

In Eq. (9) it was taken into account that $J_1(x)$ differs from zero only in the region $0 \leq x^2 \leq a^2$ where $H_z(x) < H_z^0$.

With the above formulas, the critical state equations take the following form: In the interval $0 \leq x^2 \leq b^2$ one has

$$H_0(x) = \frac{1}{2\pi} \int_0^{a^2} \frac{J_1(\sqrt{s}) ds}{s - x^2}, \quad (12)$$

and in the region $b^2 \leq x^2 \leq a^2$ we arrive at

$$H_0(x) - H_z^0 = -\frac{J_1(x)}{\gamma} + \frac{1}{2\pi} \int_0^{a^2} \frac{J_1(\sqrt{s}) ds}{s - x^2}. \quad (13)$$

In deriving Eq. (13) we have expressed $H_z(x)$ for $b^2 \leq x^2 \leq a^2$ in terms of $J_1(x)$ using the equality

$$H_z(x) = H_z^0 - \frac{J_1(x)}{\gamma} \quad (14)$$

that follows from formulas (3), (4), (6), (7). Eqs. (12), (13) are linear singular integral equations with Cauchy type kernel. The theory of such equations is well elaborated,²³ and hence we can find a , b , and $J_1(x)$ for any given H_a .

To do this, we introduce the following notations:

$$\alpha \equiv \frac{1}{\pi} \arctan \frac{\gamma}{2}, \quad \beta \equiv \frac{1}{2} - \alpha,$$

$$\alpha_+ \equiv \alpha, \quad \alpha_- \equiv \alpha + 1,$$

$$F_{\pm}(t) \equiv (a^2 - t^2)^{\alpha_{\pm}} |t^2 - b^2|^{\beta}$$

and define the function $f(t)$ by the equalities

$$f(t) = -2H_0(t), \quad 0 \leq t < b,$$

$$f(t) = 2 \sin \pi \alpha \cdot [H_z^0 - H_0(t)], \quad b < t \leq a,$$

i.e., $f(t)$ is discontinuous at $t = b$. Then, the solution of Eqs. (12), (13) can be represented as follows: In the interval $0 \leq x^2 \leq b^2$ one has

$$J_1(x) = \frac{2}{\pi} |x| F_{\pm}(x) \int_0^a \frac{f(t) dt}{(t^2 - x^2) F_{\pm}(t)}, \quad (15)$$

while in the interval $b^2 \leq x^2 \leq a^2$ we arrive at

$$J_1(x) = \cos \pi \alpha \left[f(x) + \frac{\gamma}{\pi} |x| F_{\pm}(x) \int_0^a \frac{f(t) dt}{(t^2 - x^2) F_{\pm}(t)} \right], \quad (16)$$

and $J_1(x) = 0$ for $a^2 \leq x^2 \leq w^2$. Here the integrals are taken in the sense of the Cauchy principal value; F_+ and F_- refer to positive and negative values of γ , respectively. If $\gamma < 0$, for the above solution to exist it is necessary that

$$\int_0^a \frac{f(t)}{F_-(t)} dt = 0, \quad (17)$$

and

$$\int_0^a \frac{t^2 f(t)}{F_-(t)} dt = 0. \quad (18)$$

These two equalities enable us to determine b and a when $\gamma < 0$. If $\gamma > 0$, the necessary condition for the existence of the solution is

$$\int_0^a \frac{f(t)}{F_+(t)} dt = 0. \quad (19)$$

A second relation between a and b in this case is obtained from the analysis of the magnetic field near the point $x^2 = a^2$. It turns out that

$$H_z(x) - H_z^0 \approx C_{\pm} \frac{\gamma}{2|\gamma|} (4 + \gamma^2)^{1/2} (x^2 - a^2)^{\alpha_{\pm}} \quad (20)$$

if x^2 tends to a^2 from above, and

$$H_z(x) - H_z^0 \approx C_{\pm} (a^2 - x^2)^{\alpha_{\pm}} \quad (21)$$

if x^2 approaches a^2 from below. Here C_{\pm} are certain integrals independent of x ; the subscripts $+$ and $-$ refer to the cases of positive and negative γ , respectively. Since $H_z(x) \geq H_z^0$ when $x^2 > a^2$, we find that $C_+ \geq 0$. On the other hand, one has $H_z(x) \leq H_z^0$ when $x^2 < a^2$, and thus $C_{\pm} \leq 0$. Hence, one concludes that $C_+ = 0$. This is the second equality in the case of positive γ , and it has the form

$$\int_0^b \frac{f(t) dt}{(a^2 - t^2) F_+(t)} - \frac{f(a)}{2\alpha a (a^2 - b^2)^{1/2}} + \int_b^a \left[\frac{f(t)}{t (t^2 - b^2)^{\beta}} - \frac{f(a)}{a (a^2 - b^2)^{\beta}} \right] \frac{t dt}{(a^2 - t^2)^{1+\alpha}} = 0. \quad (22)$$

The Eqs. (19) and (22) determine a and b when $\gamma > 0$.

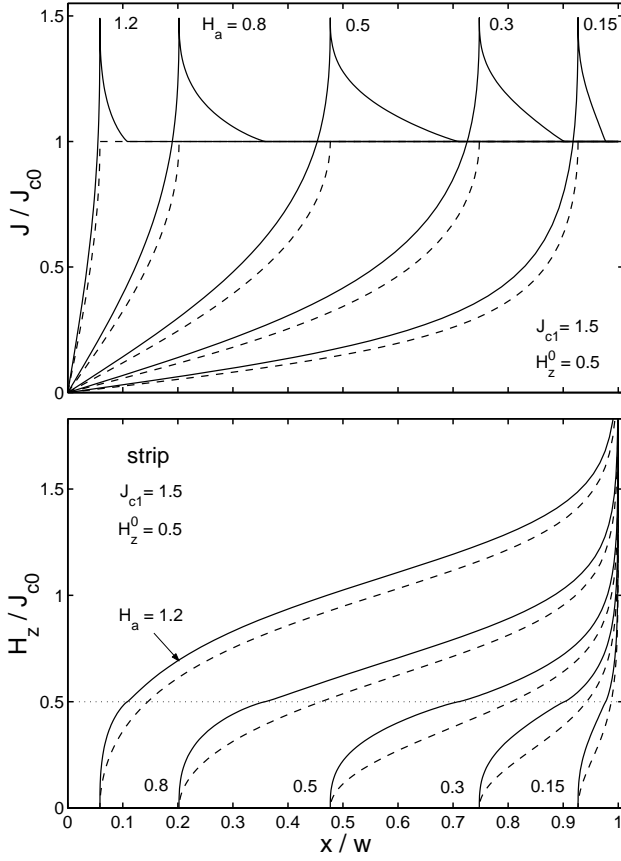


FIG. 3. Profiles of the sheet current $J(x)$ (top) and of the magnetic field $H_z(x)$ in a thin strip with width $2w$ and anisotropic pinning (solid lines) in an increasing applied field $H_a = 0.15, 0.3, 0.5, 0.8$, and 1.2 in units of $J_{c0} = 1$. The anisotropy parameters are $J_{c1}/J_{c0} = 1.5$ and $H_z^0/J_{c0} = 0.5$, thus $\gamma = 1$. The dashed lines show the profiles of an isotropic strip for the same values of the front position $b_0(H_a)$, Eq. (28). Note the sharp peak of $J(x)$ at $x = b$ of height $J(b) = J_{c1}$ and the steep front of $H_z(x)$ at $x = b$ for this type of anisotropy. At $x = a$, $J(x)$ reaches the value $J_{c0} = 1$ and $H_z(x)$ goes through the value $H_z(a) = H_z^0$ marked by a dotted line.

fitted by scaling the dependence found in the isotropic case,⁴

$$\frac{b_0}{w} = \frac{1}{\cosh(H_a/H_{cs})}, \quad (28)$$

using some effective value of H_{cs} . Rather, the shape of $b(H_a)$ essentially depends on the ratio H_z^0/H_{cs} . Therefore, measuring $b(H_a)$ in principle can give information not only on $H_{cs} = J_{c0}/\pi$ but also on H_z^0 , i.e., about the width of the peak in $j_c(\theta)$, see Eq. (5). In particular, when $H_z^0 \ll J_{c0}$, Eqs. (19), (22) lead to the following expression for the front position:

$$\left(\frac{b}{w}\right)^2 \approx \frac{1 + k \cdot (H_z^0/H_{cs})^2 \tanh^2(H_a/H_{cs})}{\cosh^2(H_a/H_{cs})}, \quad (29)$$

where the constant k is determined by the root of the equation

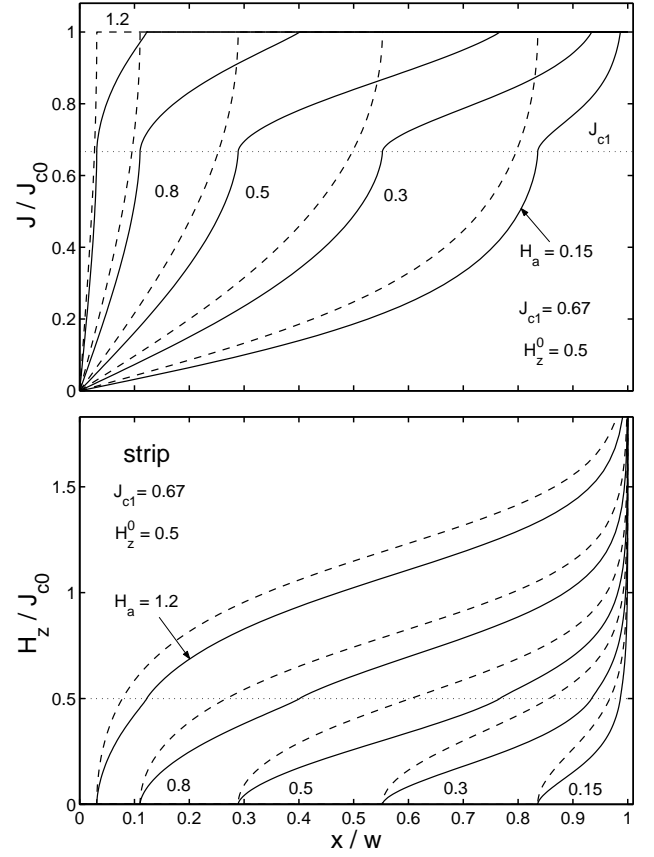


FIG. 4. As Fig. 3, but for different type of anisotropy, $J_{c1}/J_{c0} = 0.67$ and $H_z^0/J_{c0} = 0.5$, thus $\gamma = -0.67$. In this case $J(x)$ is monotonic and has an inflection point with vertical slope at $x = b$ where $J(b) = J_{c1}$ (dotted line). The penetrating front of $H_z(x)$ is now less steep than in the isotropic case, which is shown as dashed lines.

$$\frac{\pi}{4}(u^2 - 1) = u - \arctan u, \\ k = \frac{16}{\pi^2} \frac{u^2}{(1 + u^2)^2} \approx 0.394.$$

Note that the right hand side of Eq. (29) cannot be reduced to the dependence (28) in the whole interval of changes of H_a when H_z^0 is different from zero. The exact values of the front position $b(H_a)$ are shown in Fig. 7 for the limit of large $\gamma \gg 1$, for $J_{c1} = 11$ and $H_z^0 = 0 \dots 1.5$ in units of $J_{c0} = 1$.

In the third limiting case when $\gamma \rightarrow -\infty$, one has $\beta \rightarrow 1$, $C_b^- \sim -H_z^0$ and the the induction profile becomes

$$H_z(x) \propto (x^2 - b^2) \quad (30)$$

with a small prefactor of the order of $H_z^0/|\gamma|$. Thus, for $-\gamma \gg 1$ the flux front at $x = b$ practically disappears while, according to Eq. (20), the second front near $x = a$ is well developed, see Fig. 5.

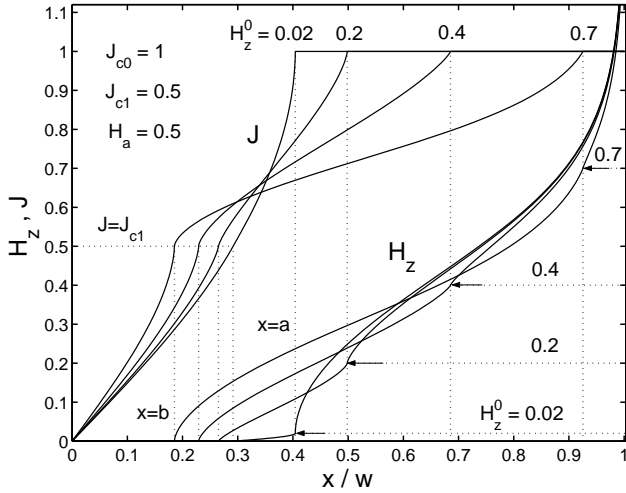


FIG. 5. Profiles of $J(x)$ (left) and $H_z(x)$ (right) in a thin strip with anisotropic pinning of the type $J_{c1} = 0.5$ for various values of $H_z^0 = 0.02, 0.2, 0.4$ and 0.7 in a constant applied field $H_a = 0.5$ (in units of $J_{c0} = 1$). The dotted lines at $x = b$, $x = a$, $J = J_{c1}$, and $H_z = H_z^0$ shall help to identify the features $J(b) = J_{c1}$, $J(a) = J_{c0}$, and $H_z(a) = H_z^0$. Note that with decreasing H_z^0 the penetrating flux front at $x = b$ becomes less pronounced and a new front appears at $x = a$. In the limit $H_z^0 \rightarrow 0$ only the front at $x = a$ remains and the profiles look like in the isotropic strip with b replaced by a .

Finally, we consider in some detail the case of small negative values of γ when $H_z^0 \gg H_{cs} = J_{c0}/\pi$ while the ratio J_{c0}/J_{c1} is not close to unity. This case can give some idea of pinning by columnar defects, which produce a peak in $j_c(\theta)$ at $\theta = 0$. Indeed, if one assumes that the characteristic width of the peak, θ_0 , is small ($\theta_0 \ll 1$), then it follows from the definitions of H_z^0 and γ that $H_z^0 \approx J_{c0}/2\theta_0$ and $|\gamma| < 2\theta_0$. Since the solution with $\gamma = 0$ and $J_c = J_{c1}$ describes the critical state in the strip before the irradiation [we assume that the columnar defects do not change $j_c(\theta)$ at $\theta > \theta_0$], the difference between the solutions corresponding to $\gamma \neq 0$ and $\gamma = 0$ provides information on pinning by columnar defects. In the considered case this difference is small, and it can be analyzed analytically. In particular, we obtain the following relation between the positions of the flux fronts, b and b_1 , obtained at the same H_a in the strip with and without columnar defects, respectively:

$$\operatorname{arccosh} \frac{w}{b_1} - \operatorname{arccosh} \frac{w}{b} = \frac{|\gamma|}{\pi} g(h), \quad (31)$$

where $h \equiv \pi H_a/J_{c1}$, $w/b_1 = \cosh(h)$, and the function $g(h)$ has the form:

$$g(h) = \int_0^h \ln(2 \cosh t) dt. \quad (32)$$

Since g is a nonlinear function of h ,

$$g(h) \approx \frac{1}{2}h^2 + 0.411(1 - e^{-1.8h}), \quad (33)$$

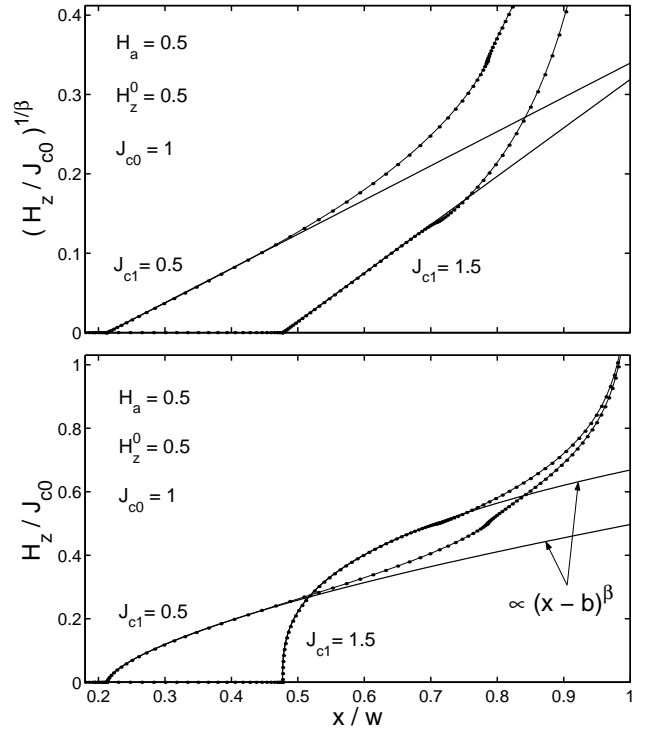


FIG. 6. Comparison of the shape of the profile $H_z(x)$ near the flux front with the expression $(x - b)^\beta$ suggested by Eq. (26). Shown are examples with $H_a = 0.5$ (in units of $J_{c0} = 1$) and two different anisotropies: $H_z^0 = 0.5$, $J_{c1} = 1.5$ (thus $\gamma = 1$, $\alpha = 0.148$, $\beta = 0.352$, $b = 0.477$, $a = 0.711$) and $H_z^0 = 0.5$, $J_{c1} = 0.5$ (thus $\gamma = -1$, $\alpha = -0.148$, $\beta = 0.648$, $b = 0.213$, $a = 0.784$). The exact $H_z(x)$ (dotted lines) is well fitted over a large interval of x by the function $c \cdot (x - b)^\beta$ (solid lines) with $c = 0.840$ or $c = 0.580$ for these two examples (with x and b in units of the strip half width w). The solid lines in the upper plot are straight lines fitting $H_z(x)^{1/\beta}$.

the exact dependence $b(H_a)$ can not be described by Eq. (28) with some effective H_{cs} . The prefactor

$$\frac{|\gamma|}{\pi} \approx \frac{2\theta_0}{\pi} \frac{j_c(0) - j_c(\pi/2)}{j_c(0)}$$

in Eq. (31) is determined by the characteristics of pinning by the columnar defects, i.e., by the width and height of the peak in $j_c(\theta)$.

IV. CONCLUSIONS

An exact solution of the critical state equations for the strip in perpendicular magnetic field is derived for an induction-dependent critical sheet current $J_c(H_z)$ described by Eqs. (4). This model dependence may be used to simulate the intrinsic pinning by CuO planes

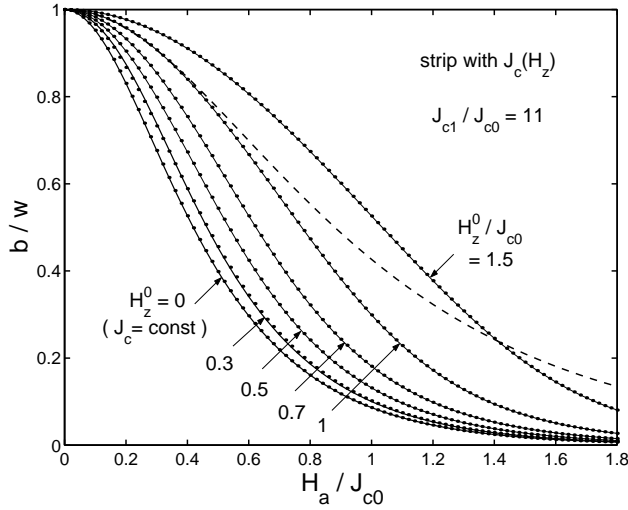


FIG. 7. The position b of the flux front, or penetration depth $w - b$, of a superconductor thin strip with width $2w$ and various $J_c(H_z)$ dependences, Eq. (4), plotted versus the applied magnetic field H_a in units of $J_{c0} = 1$. The dotted lines are computed as described in Sct. III for anisotropy parameters $J_{c1} = 11$ and $H_z^0 = 0, 0.3, 0.5, 0.7, 1$, and 1.5 . The bold solid lines for $H_z^0 = 0$ and 0.3 are from Eq. (29) and fit the exact data very well. The dashed line $b/w = 1/\cosh[H_a/(2.1H_{cs})]$, obtained by stretching the isotropic ($J_c = \text{const}$) expression, Eq. (28), by a factor of 2.1, demonstrates that such scaling of the isotropic result cannot fit the anisotropic result.

($\gamma > 0$) or pinning by extended defects ($\gamma < 0$) in high- T_c superconductors. In the case $\gamma > 0$, the H_z profile in the vicinity of the flux front is sharper than in the isotropic case, and the current density has a sharp peak there. In the limiting case, $\gamma \gg 1$, which may describe the intrinsic pinning in high- T_c superconductors, the field profile $H_z(x)$ has a sharp rectangular step. In the opposite situation, $\gamma < 0$, *two flux fronts* can occur in the superconductor; the H_z profile near $x = b$ is less steep than in the isotropic case, and the current density is a monotonic function of x . In both cases of positive and negative γ the profile $H_z(x)$ in a sufficiently large vicinity of the flux front is well approximated by the expression $H_z(x) \approx (x - b)^\beta$ with the exponent $\beta = 0.5 - \pi^{-1} \arctan(\gamma/2)$.

The experimental investigation of flux-density profiles near the flux front and of the H_z dependence of the penetration depth can give information on the strength and anisotropy of flux line pinning in superconductors.

ACKNOWLEDGMENTS

G.P.M. acknowledges the hospitality of the Max-Planck-Institut für Metallforschung, Stuttgart.

APPENDIX A: NUMERICAL EVALUATION

The condition that two integrals have to vanish, e.g. Eqs. (17,18) of the form $I_1(a, b) = 0$ and $I_2(a, b) = 0$, we satisfy by minimizing the function $U(a, b) = I_1^2 + I_2^2$ with respect to a and b . After this we calculate the sheet current $J_1(x)$ from Eqs. (15,16) and the magnetic field $H_z(x)$ from Eqs. (9) and (14).

The integrals (9), (15-19), and (22) over the variable t have integrands which possess one or several infinities at the points $t = 0$, $t = x$, $t = b$ and $t = a$ where the denominators vanish. We evaluate such integrals in the following way.

In the integrals containing a factor $(t - x)^{-1}$ we subtract the singular part and integrate it analytically, e.g.,

$$\int_0^a \frac{f(t) dt}{t^2 - x^2} = \int_0^a \frac{f(t) - f(x)}{t^2 - x^2} dt - \frac{f(x)}{2x} \ln \frac{a+x}{a-x}. \quad (\text{A1})$$

Then we divide the integration interval into pieces bounded by the remaining singularities, $0 \leq t \leq b$, $b \leq t \leq a$, and $a \leq t \leq 1$. In each interval we substitute the integration variable by an appropriate function $t = t(u)$ and integrate over u such that the new integrand has no infinity and vanishes rapidly at the boundaries. This new integral may thus be evaluated as a sum over an equidistant grid u_i with constant weights. For example we write

$$\int_0^\tau g(t) dt = \int_0^1 g[t(u)] t'(u) du \approx \sum_{i=1}^N g_i w_i \quad (\text{A2})$$

with $g_i = g[t(u_i)]$, $u_i = (i - 1/2)/N$, $w_i = t'(u_i)/N$, $t'(u) = dt/du$, and $i = 1, 2, 3, \dots, N$. This integration method is very accurate if the substitution is chosen such that the weights w_i and the products $g_i w_i$ vanish rapidly at the integration boundaries, e.g., $w_i \sim u_i^p$ and $w_i \sim (1 - u_i)^q$ with $p \gg 1$ and $q \gg 1$. Simple choices of this substitution in the example (A2) are

$$t(u) = (3u^2 - 2u^3)\tau, \quad t'(u) = 6u(1 - u)\tau, \quad (\text{A3})$$

or better,

$$t(u) = (10u^3 - 15u^4 + 6u^5)\tau, \quad t'(u) = 30u^2(1 - u)^2\tau. \quad (\text{A4})$$

Higher accuracy is achieved by the following substitution. We chose equidistant $u_i = (i - 1/2)/N$ as above and then iterate (A3) m times starting with $s_i = u_i$ and $w_i = \tau/N$ according to

$$w := 6(s - s^2)w, \quad s := 3s^2 - 2s^3 \quad (m \text{ times}). \quad (\text{A5})$$

Finally we write $t(u_i) = s_i\tau$. The weights $w_i = t'(u_i)/N$ of this substitution vanish at the boundaries with exponents $p = q = 2^{(m-1)}$, which can be made arbitrarily large. For example, using $m = 5$ iterations one gets the exponents $p = q = 2^4 = 16$.

An infinity $g(t) \propto 1/t^\eta$ in the original integral (A2) leads, after this substitution, to a new integrand vanishing at $t = 0$ as $g[t(u)]t'(u) \propto u^\vartheta$ with $\vartheta = p(1 - \eta) - \eta$.

Thus, for the example $\eta = 1/2$ with $p = 16$ the new integrand near $u = 0$ vanishes as $u^{7.5}$ and the terms in the sum (A2) as $(i - 1/2)^{7.5}$, in spite of the singular original integrand. For general exponent η , to reach high accuracy one should choose m so large that the new exponent is $\vartheta = (1 - \eta)2^{m-1} - \eta \geq 4$, or approximately $m \geq 3.5 - 1.5 \ln(1 - \eta)$. To avoid spurious results due to rounding errors, one has to add in all vanishing denominators a small $\epsilon \approx 10^{-15}$ by writing, e.g., $(|t^2 - b^2| + \epsilon)^\beta$.

In the limit of a large negative slope $\gamma \rightarrow -\infty$ one has $\beta \rightarrow 1$ and the integrals (17,18) containing a factor $|t^2 - b^2|^{-\beta}$ are close to diverging. In this case the singular part in these integrals should be integrated analytically, similar as shown in Eq. (A1). The subtracted terms are conveniently chosen such that the integral which has to be taken analytically is simple, e.g., $\int t \cdot (b^2 - t^2)^{-\beta} dt$. Note that the numerator $f(t)$ in Eqs. (15–19,22) is discontinuous at $t = b$.

¹ C. P. Bean, Phys. Rev. Lett. **8**, 250 (1962); Rev. Mod. Phys. **36**, 31 (1964).

² A. M. Campbell and J. E. Evetts, Adv. Phys. **72**, 199 (1972).

³ P. N. Mikheenko and Yu. E. Kuzovlev, Physica C **204**, 229 (1993).

⁴ E. H. Brandt, M. Indenbom, and A. Forkl, Europhys. Lett. **22**, 735 (1993);

⁵ W. T. Norris, J. Phys. D **3**, 489 (1970).

⁶ G.P.Mikitik and E.H. Brandt, Phys. Rev. B **60**, 592 (1999).

⁷ E. H. Brandt, Phys. Rev. B **54**, 4246 (1996).

⁸ E. H. Brandt, Phys. Rev. B **58**, 6506, 6523 (1998).

⁹ Y. B. Kim, C. F. Hempstead, and A. Strnad, Phys. Rev. **129**, 528 (1963).

¹⁰ S. Senoussi, J. Physique III **2**, 1041 (1992).

¹¹ E. H. Brandt, Rep. Prog. Phys. **58**, 1465 (1995).

¹² J. McDonald and J. R. Clem, Phys. Rev. B **53**, 8643 (1996).

¹³ D. V. Shantsev, Y. M. Galperin, and T. H. Johansen, Phys. Rev. B **60**, 13112 (1999).

¹⁴ E. H. Brandt, Physica C **235-240**, 2939 (1994).

¹⁵ I. M. Babich and G. P. Mikitik, Phys. Rev. B **54**, 6576 (1996).

¹⁶ I. M. Babich and G. P. Mikitik, Pis'ma Zh. Eksp. Teor. Fiz. **64**, 538 (1996) [JETP Lett. **64**, 586 (1996)];

¹⁷ I. M. Babich and G. P. Mikitik, Phys. Rev. B **58**, 14207 (1998).

¹⁸ G. P. Mikitik and E. H. Brandt, Phys. Rev. B **61** (submitted).

¹⁹ E. Zeldov, A. I. Larkin, V. B. Geshkenbein, M. Konczykowski, D. Majer, B. Khaykovich, V. M. Vinokur, and H. Strikman, Phys. Rev. Lett. **73**, 1428 (1994); M. V. Indenbom and E. H. Brandt, Phys. Rev. Lett. **73**, 1731 (1994); I. L. Maksimov and A. A. Elistratov, Pis'ma Zh. Eksp. Teor. Fiz. **61**, 204 (1995) [Sov. Phys. JETP Lett. **61**, 208 (1995)]; N. Morozov, E. Zeldov, D. Majer, and B. Khaykovich, Phys. Rev. Lett. **76**, 138 (1996); M. Benkraouda and J. R. Clem, Phys. Rev. B **53**, 5716 (1996); Phys. Rev. B **58**, 15103 (1998); C. J. van der Beek, M. V. Indenbom, G. D'Anna, and W. Benoit, Physica C **258**, 105 (1996). R. Labusch and T. B. Doyle, Physica C **290**, 143 (1997); T. B. Doyle, R. Labusch, and R. A. Doyle, Physica C **290**, 148 (1997); E. H. Brandt, Phys. Rev. B **59**, 3369 (1999); Phys. Rev. B **60**, 11939 (1999).

²⁰ E. H. Brandt and M. V. Indenbom, Phys. Rev. B **48**, 12893 (1993).

²¹ E. Zeldov, J. R. Clem, M. McElfresh and M. Darwin, Phys. Rev. B **49** 9802, (1994).

²² M. Tachiki and S. Takahashi, Solid State Commun. **70**, 291 (1989).

²³ N. Muskhelishvili, *Singular integral equations*, P. Nordhoff, Groningen, Holland (1953).

²⁴ B. Roas, L. Schultz, and G. Saemann-Ischenko, Phys. Rev. Lett. **64**, 479 (1990).